

Rigidity of Wonderful Compactifications under Fano Deformations

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DEF: X sm. proj. is rigid if \forall sm family $\mathcal{X} \xrightarrow[\pi]{}$
with (i) B connected
(ii) every fiber is projective
& one fiber $\mathcal{X}_0 \cong X$ then $\mathcal{X}_b \cong X \ \forall b.$

Remark:

By Ehresmann, $\mathcal{X}_b \xrightarrow{\text{diff.}} \mathcal{X}_0 \cong X$

Example:

(1) \mathbb{P}^1 is rigid

(2) Any sm proj. curve of genus ≥ 1 is NOT rigid.

(3) \mathbb{P}^n is rigid.

[Hizebruch-Kodaira-Yau] $X = \text{compt Kähler}$

if X is homeom. to \mathbb{P}^n , then $X \cong \mathbb{P}^n$

idea:

$$1 = b_{2i}(\mathbb{P}^n) = b_{2i}(X) = \sum h^{p, 2i-p}(X)$$

$$\text{so } h^{p,q}(X) = \begin{cases} 1 & 0 \leq p = q \leq n \\ 0 & \text{other} \end{cases}$$

$$\rightsquigarrow \text{Pic}(X) \cong \mathbb{Z} L$$

$$f: X \xrightarrow{\text{hom}} \mathbb{P}^n \quad \text{Pic}(\mathbb{P}^n) \xrightarrow[\text{isom}]{} f^* \text{Pic}(X)$$

either

$$f^*(\mathcal{O}(1)) = L \Rightarrow \text{index}(X) = n+1$$

$$\Rightarrow X \cong \mathbb{P}^n \quad (\text{Kobayashi-Ochiai})$$

or

$$f^*(\mathcal{O}(1)) = -L \xrightarrow{\text{Yau}} X \text{ is a ball quotient}$$

$$\text{then } \pi_1(X) \not\simeq 1$$

Rmk: (1) Brieskorn proved this for \mathbb{Q}^{2n}

(2) Kähler condition is essential (o.k. $n \leq 2$)

Assume S^6 has a complex str. $X = Bl_o(S^6)$
(not Kähler as $b_2 = 0$)

then $X \overset{\text{diff}}{\sim} S^6 \# \overline{P^3} \overset{\text{diff}}{\sim} \overline{P^3} \overset{\text{diff}}{\sim} P^3$

$E \subset X$ except divisor

$$c_1(S^6) = 0 \quad \text{so} \quad c_1(X) = -2E$$

$$\text{then } c_1(X)^3 = -8 \neq c_1(P^3)^3 = 4^3$$

(3) if drop projective condition in DEF

then much harder.

Siu proved rigidity of P^n

Hwang: \mathbb{Q}^n

No other results in this set-up.

Question What are next var. to check rigidity?

Observation: X rigid \Rightarrow locally rigid, $\mathcal{X}_b \simeq X$
for b near $o \in B$

easier to check:

$H^1(X, T_X) = 0 \Rightarrow X$ is loc. rigid.

equivalent if $H^2(X, T_X) = 0$

O.K. if X is Fano by Kodaira-Nakano

[Bott] G/P is loc. rigid

[Bien-Brion] Fano regular G -varieties

[Pasquier-Perrin] Some 2-orbit varieties

[Bai-Fu-Manivel] Some complete inter. in G/P

Example $\mathbb{P}^1 \times \mathbb{P}^1$ is NOT rigid

$$\mathbb{P}^1 \times \mathbb{P}^2 \supset X_\lambda = \{s^{2a}x - t^{2a}y - \lambda s^a t^a z = 0\}$$

then $X_\lambda \simeq \mathbb{P}^1 \times \mathbb{P}^1$ if $\lambda \neq 0$

$$X_0 \simeq \mathbb{P}(\mathcal{O}(-a) \oplus \mathcal{O}(a))$$

Hirzebruch surface of d^a 2a

$\mathbb{P}^1 \times \mathbb{Q}^1 \subset \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ can deform to $\mathbb{S}_{1,3}$ rational scroll

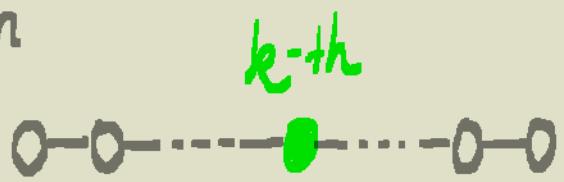
$$\mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(3))$$

this is the only poss. in \mathbb{P}^5

by class. of deg 4 subvar.

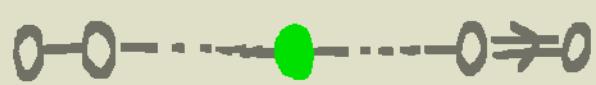
G/P of $b_2=1 \Leftrightarrow G$ simple & $P \subset G$ maxi.
 $\Leftrightarrow G$ simple + a simple root
 i.e. marked Dynkin diagram

type A_n



$$Gr(k, n+1) = A_n / P_k$$

type B_n



$$OG(k, 2n+1) = B_n / P_k$$

e.g. $O - \bullet \not\rightarrow O$

$$B_3 / P_2$$

$\{$ lines on $Q^5 \subset \mathbb{P}^6 \}$

$$\dim(B_3 / P_2) = 7$$

[Hwang-Mok] G/P with $b_2=1$ is rigid

$\Leftrightarrow G/P$ is not of type B_3/P_2

Rmk

(1) Pasquier-Perrin constr. explicit def. of B_3/P_2

X
 \downarrow

$$X_t = \overline{G_2 \cdot [z_3 + (z_0 \wedge z_3 + z_1 \wedge z_2)t + z_1 \wedge z_3]} \subset \mathbb{P}(V \oplus V_2)$$

Δ

$$X_t \cong B_3/P_2 \text{ if } t \neq 0$$

X_0 is not homogeneous

(2) Hwang-Li: this is the only possibility!

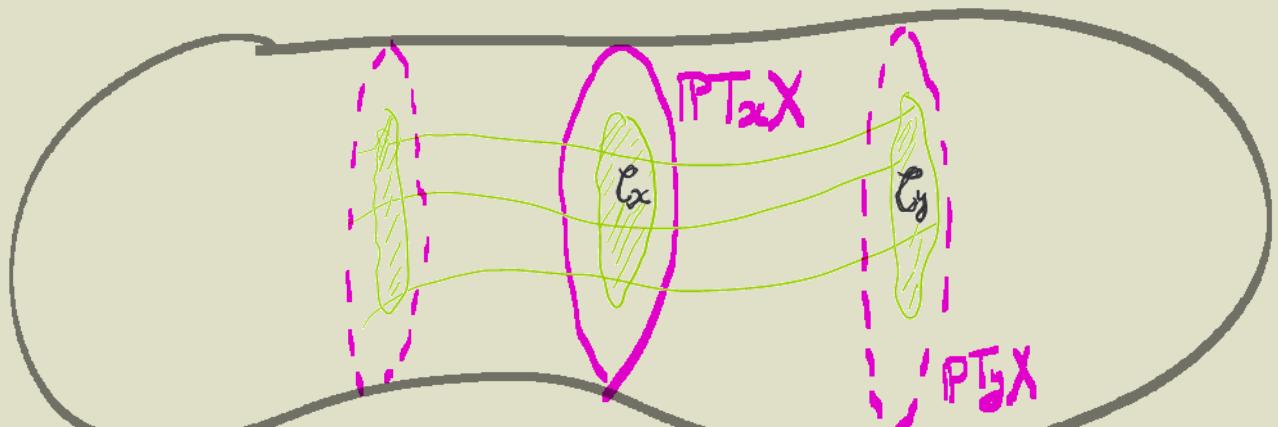
(3) $b_2=1$ is essential to this approach.

Key ingredients: VMRT theory of Hwang-Mok

- × sm proj covered by rat. curves
- ✗ a family of min. rat. curves (e.g. lines)
- $x \in X$ general

$$X_x = \{[C] \in X \mid x \in C\} \dashrightarrow C_x \subset PT_x X$$

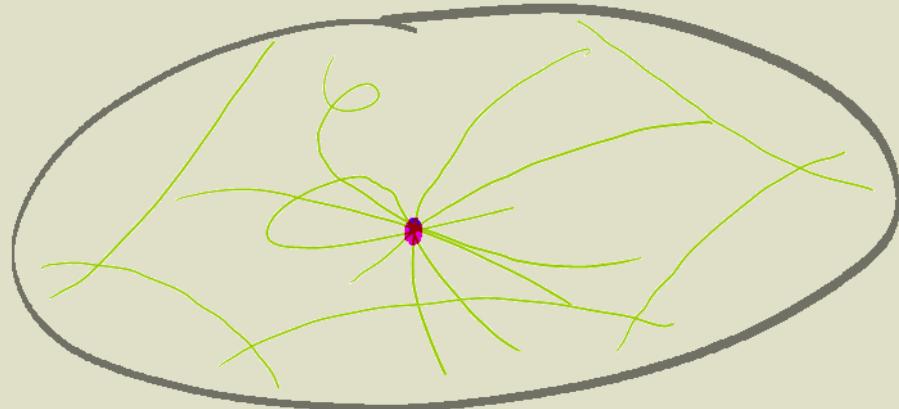
$$[C] \longrightarrow [T_x C]$$



PTX



X



Example: VMRT of \mathbb{G}/\mathbb{P} with $b_2=1$

$\exists!$ minimal G -equiv. embedding $\mathbb{G}/\mathbb{P} \hookrightarrow \mathbb{P}^N$
& \mathbb{G}/\mathbb{P} is covered by lines.

e.g.

$$\mathrm{Gr}(k, n+1) \hookrightarrow \mathbb{P} \wedge^k \mathbb{C}^{n+1} \quad \text{Pl\"ucker}$$

$$\begin{array}{c} k^{\text{th}} \\ 0 \cdots 0 \textcolor{red}{\bullet} 0 \cdots 0 \end{array} \xrightarrow{\text{VMRT}} \begin{array}{c} \text{VMRT} \\ \textcolor{red}{\bullet} \textcolor{red}{\bullet} \cdots \textcolor{red}{\bullet} \\ \mathbb{P}^{k-1} \times \mathbb{P}^{n-k} \end{array}$$

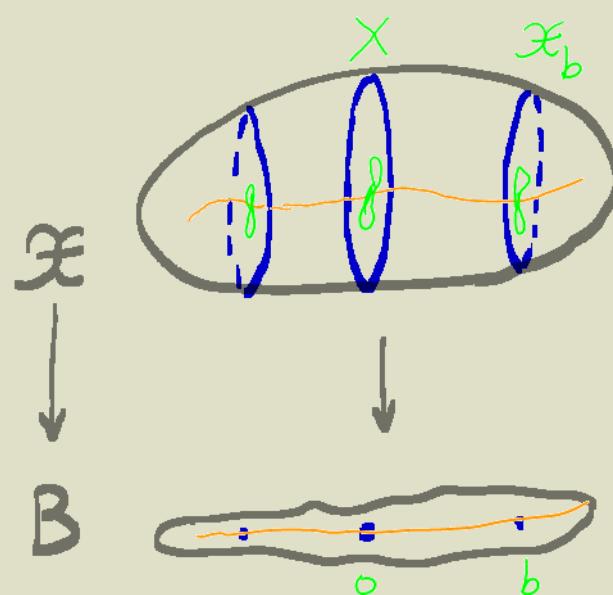
$$B_3/\mathbb{P}_2 = \{P^1 \text{ on } Q^5\} \subseteq \mathrm{Gr}(2, 7) \hookrightarrow \mathbb{P} \wedge^2 \mathbb{C}^7 = \mathbb{P}^{20}$$

$$\begin{array}{c} \text{VMRT} \\ \textcolor{red}{\bullet} \Rightarrow \bullet \end{array} \xrightarrow{\text{VMRT}} \begin{array}{c} \bullet \quad \bullet \\ \mathbb{P}^1 \times \mathbb{Q}^1 \end{array}$$

Rmk. VMRT of \mathbb{G}/\mathbb{P} is homog. for long root
quasi-homog. for short root

Idea of Hwang-Mok's proof

(1) Invariance of VMRT



x_b & x_0 have same
VMRT at general points

(2) Recognition G/P from VMRT

[Mok, Hong-Hwang, Hwang-Li, Hwang-Li-Timashov]

IHSS long roots C_n

Z sm proj. $b_2=1$ with same VMRT as G/P

$$\Rightarrow Z \cong G/P$$

Rmk:

B_3/P_2 has VMRT $P^1 \times Q^1 \hookrightarrow P^1 \times P^2 \hookrightarrow P^5 \hookrightarrow P^6$

which has deform to $P(0(0)\oplus 0(2))$
step (1) fails.

Higher Picard number: Fano deformations

$\mathbb{P}^1 \times \mathbb{P}^1$ is not rigid, deform to $\mathbb{P}(O(-a) \oplus O(a))$
if $a \neq 0$, $\not\cong$ not Fano

Rigidity under Fano deform.

$$\begin{array}{c} X \\ \pi \downarrow \\ B \end{array}$$

X_b is Fano
 $\forall b \in B$

(Same if $b_2(X)=1$)

[Weber-Wisniewski]

G/B is rigid under Fano deform.

[Q. Li]

(1) $X \times Y$ is rigid under Fano deform

\Leftrightarrow so are X & Y

(2) many G/P are rigid under Fano deform.

Ex: $\mathbb{P}\mathcal{T}_{\mathbb{P}^{2n+1}}$ is not rigid under Fano deform.

$\rightsquigarrow \mathbb{P}(N(-1) \oplus O)$ $N = \text{null-correlation bdle}/\mathbb{P}^{2n+1}$

Wonderful compactifications

$G = \text{semisimple adjoint Lie gp}$ (e.g. PSL)

[De Concini - Procesi]

$\exists!$ Fano sm $G \times G$ -variety X (or \bar{G}) s.t.

(1) $X \supset O \stackrel{\text{open}}{\simeq} G \times G / \text{diag}(G) \simeq G$

(2) $\partial X = X \setminus O = \bigcup_{i=1}^l D_i$ D_i sm div. with n.c.
 $l = \text{rk}(G)$

(3) $G \times G$ -orbits in $X \xleftarrow{1-1} D_{i_1} \cap \dots \cap D_{i_j}$

Rmk

(i) if $G = G_1 \times \dots \times G_n$ then $\bar{G} = \bar{G}_1 \times \dots \times \bar{G}_n$

(ii) $\overline{\text{PSL}_2} = \mathbb{P}^3$

$\text{PSL}_2 \hookrightarrow \mathbb{P}\text{End}(\mathbb{C}^2) = \mathbb{P}^3$ boundary $\simeq \mathbb{P}^1 \times \mathbb{P}^1$

(iii) $\text{Pic}(X) \cong \bigoplus_{i=1}^l \mathbb{Z} D_i$ so $\rho_X = l = \text{rk}(G)$

Construction of wond. comp.

λ regular dominant weight V_λ irred. rep.

$$G \hookrightarrow \mathrm{PGL}(V_\lambda) \hookrightarrow \mathrm{PEnd}(V_\lambda) \xleftarrow{\text{open}} G \times G$$

$X = \overline{G}$ closure of G in $\mathrm{PEnd}(V_\lambda)$
(indep. of choice of λ)

Classical constr. of complete collinearations

V = vect. space $\dim = l+1$

$$\mathrm{PSL}(V) \subset \mathrm{PEnd}(V) \supset Z_l \supset Z_{l-1} \supset \dots \supset Z_1 \cong \mathrm{PV} \times \mathrm{PV}^*$$

$$Z_i = \{ \mathrm{rk} \leq i \}$$

$\overline{\mathrm{PSL}(V)}$ = successive blowup of $\mathrm{PEnd}(V)$
along strict transf. of Z_i ($i = l \dots l+1$)

$$= \text{closure of image } \mathrm{PEnd}(V) \dashrightarrow \bigcup_{i=0}^{l+1} \mathrm{PEnd}(\lambda^i V) \\ f \mapsto (\lambda^i f)_i$$

Main theorem (Fu-Li)

Wonderful comp. of G is rigid under Fano deform.

Rmks

- (1) may assume G simple by Li's result.
- (2) \overline{G} is locally rigid by Bien-Brion.
(this answers a question of Bien-Brion)
- (3) We suspect \overline{G} is rigid in the stronger sense.
- (4) may reduce to

$$\begin{array}{c} \mathcal{X} \\ \pi \downarrow \\ o \in \Delta \end{array}$$

$$\mathcal{X}_t \cong \overline{G} \quad \forall t \neq 0$$

\mathcal{X}_0 is Fano

$$\Rightarrow \mathcal{X}_0 \cong \overline{G}$$

Show

Minimal rational curves on \overline{G}

[Brion-Fu] G simple

(i) $\exists!$ family of min rat. curves on \overline{G} .

(ii) VMRT $\mathcal{C}_x \simeq G/P_\theta \subset \mathbb{P} T_x \overline{G} \cong \mathbb{P}^9$ if $G \neq A_2$
unique closed G -orbit

(iii) $G = A_2$ then $\mathcal{C}_x \simeq \mathbb{P}^l \times \mathbb{P}^l \hookrightarrow \mathbb{P}^9$

Rmks

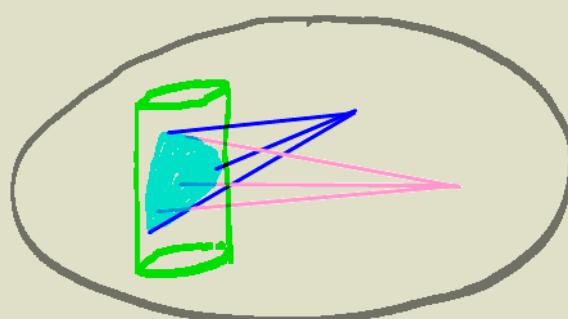
(1) $l \geq 2$ then \overline{G} is not covered by lines

(2) $\overline{\mathrm{PSL}(V)}$:

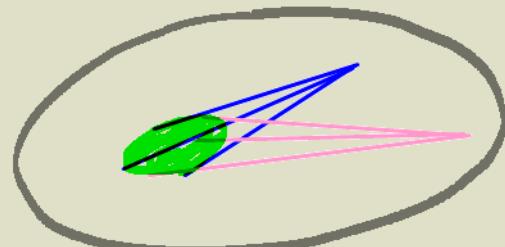
min. rat. curves

||
strict transf. of lines

meeting $Z_1 = \mathbb{P}V \times \mathbb{P}V^*$



↓ Blowups



\mathbb{G}/P_θ : Homog. Fano contact mfd



$$\mathbb{P} \mathbb{T}_{\mathbb{P}^l}^*$$



$$\mathbb{P}^{2l-1}$$



5-dim.



15-dim.



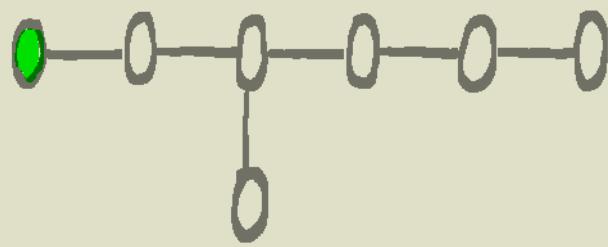
$$B_2/P_2 = \text{Gr}_0(2, 2l+1)$$



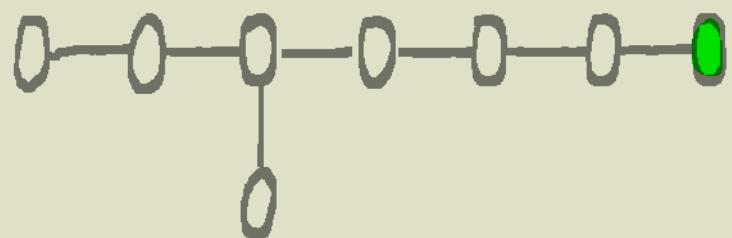
$$D_l/P_2 = \text{Gr}_0(2, 2l)$$



21-dim.



dim=33



dim=57

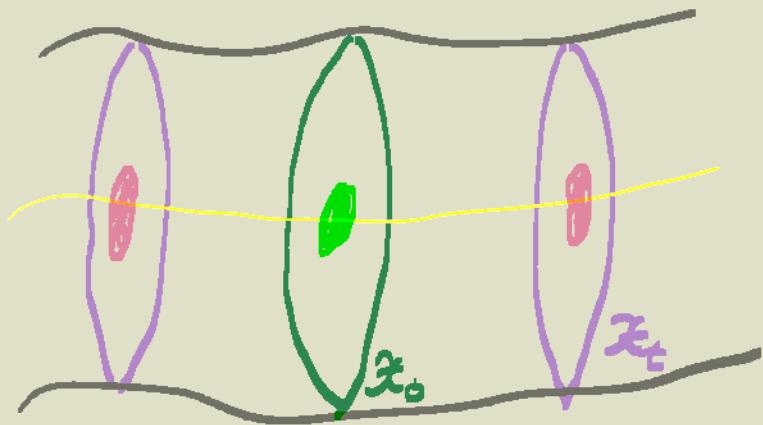
Rmk.: \mathbb{G}/P_θ is homog. of $b_2=1$ except type A

[Hwang] \mathbb{G}/P_θ is rigid except type B_3

Invariance of VMRT

$$\mathcal{X} \downarrow \Delta$$

$$\mathcal{X}_t \simeq \overline{G} \quad t \neq 0$$



$\sigma: \Delta \rightarrow \mathcal{X}$ local section

$$\downarrow \pi$$



$$\mathcal{X}_\sigma = \bigcup_t \mathcal{X}_{\sigma(t)}$$

smooth family $\mathcal{X}_{\sigma(t)} \simeq G/P_t \quad t \neq 0$

$$\downarrow \Delta$$

by Hwang $\mathcal{X}_{\sigma(0)} \simeq G/P_0$

except $G = B_3$

then study $\mathcal{X}_{\sigma(0)} \rightarrow \mathcal{L}_{\sigma(0)} \subset \text{PG}$ to show invariance
 ↑
 normalisation
 turns out to be isom.

Invariance for type A

$$\overline{A}_e = \overline{\text{PSL}(V)}$$

$\phi \downarrow$ successive blowups
given by $|L|$

$$\text{PSL}(V) \subset Z = \text{PEnd}(V) > Z_2 > \dots > Z_1$$

↑
VMRT

$$\begin{array}{ccc} X & \xrightarrow{\Phi_L} & Z \\ \pi \downarrow & & \swarrow \\ \Delta & & \end{array}$$

$$B = \text{closure of } Z_1 \times \Delta^*$$

$$\begin{array}{ccc} \mathcal{E}_x & \xrightarrow{\cong} & \rho_x(B_0) \\ \downarrow & & \downarrow \\ \text{PT}_x \mathcal{X}_0 & \longrightarrow & \text{PT}_x Z_0 \end{array}$$

Invariance of type B_3

$$\mathcal{X} \supset \mathcal{X}_t \simeq \overline{B}_3$$

$t \neq 0$

VMRT
along a section

$$\mathcal{C}_0 \quad \mathcal{C}_{0(t)} \simeq B_3/P_2$$

$t \neq 0$

$\left\{ \begin{array}{l} \text{VMRT} \\ \Delta \end{array} \right.$

$$L_t \simeq \mathbb{P}^1 \times Q^1 \quad t \neq 0$$

$$L \downarrow \Delta$$

$$L_0 \subset \mathbb{P}^5 \quad \text{deg} = 4$$

two poss. $\mathbb{P}^1 \times Q^1$ or $S_{1,3} = \mathbb{P}(0100013)$

Ruled out by considering
foliations of \mathbb{P}^1 & Q^1
on gener. fibers.

if $L_0 \simeq \mathbb{P}^1 \times Q^1$

then $\mathcal{C}_{0(0)} \simeq B_3/P_2$

[recognition by VMRT]

Recognition fails in general

G/P with $b_2=1$ is determined by its VMRT
but fails in general

[Fu-Hwang] $\text{Lag}(2n) \cap H$ sm has same VMRT
 $b_2=1$ τ has moduli

[BFM] S_5 has two codim 2 sections

$b_2=1$ smooth, non isom., quasi-homog.
with same VMRT

Special $S_5 \cap H^2$ is not loc. rigid

general is loc. rigid, but not rigid

Many examples with $b_2 > 1$ by blow-ups.

New idea: local rigidity of \mathfrak{X}_0

If \mathfrak{X}_0 is loc. rigid, then $\mathfrak{X}_0 \simeq \mathfrak{X}_t \simeq \bar{G}$

$$\uparrow$$
$$H^1(\mathfrak{X}_0, T_{\mathfrak{X}_0}) = 0$$

Kodaira-Nakano $H^p(\mathfrak{X}_0, \Omega_{\mathfrak{X}_0}^q \otimes \mathcal{L}) = 0$ if $p+q > n$

\mathfrak{X}_0 Fano $\Rightarrow H^i(\mathfrak{X}_0, T_{\mathfrak{X}_0}) = H^i(\mathfrak{X}_0, \Omega_{\mathfrak{X}_0}^{n-1} \otimes K_{\mathfrak{X}_0}^{-1}) = 0$
for $i \geq 2$

$$h^0(T_{\mathfrak{X}_0}) - h^1(T_{\mathfrak{X}_0}) = \chi(T_{\mathfrak{X}_0}) = \chi(T_{\mathfrak{X}_t}) = 2 \dim G$$

$$\text{so } h^1(T_{\mathfrak{X}_0}) = h^0(T_{\mathfrak{X}_0}) - 2 \dim G$$

need to bound $h^0(T_{\mathfrak{X}_0})$

Bound Automorphism from VMRT

X sm. proj. uniruled $\dim n$ with sm VMRT

then $\text{aut}(X) \leq n + \text{aut}(\hat{C}_X) + \text{aut}(\hat{G}_X)^{(1)}$

aut of VMRT prolongation

equality $\Rightarrow X$ is equiv. compt. of \mathbb{C}^n

[Fu-Hwang] classify $\text{aut}(\hat{G}_X)^{(1)} \neq 0$

Application

[Fu-Ou-Xie] X Fano $b_2=1$ $\dim n$ nice VMRT

irred sm non-deg

then $\text{aut}(X) \leq \frac{n(n+1)}{2}$ except $\mathbb{P}^n, \mathbb{Q}^n, \text{Gr}(2,5)$

equality $\Leftrightarrow X \simeq \text{Lag}(6) \cap H, \text{Gr}(2,5) \cap H$

Bound $h^0(\mathcal{X}_0, T_{\mathcal{X}_0})$

$$h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) = \text{aut}(\mathcal{X}_0) \leq \dim G + \text{aut}(\hat{\mathcal{C}}_x)$$

if not of type C

(type C, $\mathcal{C}_x \simeq \mathbb{P}^{2l-1}$ with $\text{aut}(\hat{\mathcal{C}}_x)^{(1)} \neq 0$)

$$\begin{aligned}\text{aut}(\hat{\mathcal{C}}_x) &= \text{aut}(\mathcal{C}_x) + 1 = \text{aut}(G/P_0) + 1 \\ &= \dim G + 1\end{aligned}$$

hence if $G \neq C_x$

$$\text{then } h^0(T_{\mathcal{X}_0}) \leq 2\dim G + 1 - 2\dim G = 1$$

either $h^0(T_{\mathcal{X}_0}) = 0$, so $\mathcal{X}_0 \simeq \overline{G}$

or $h^0(T_{\mathcal{X}_0}) = 1$, $\mathcal{X}_0 = \text{equiv. comp. of } \mathbb{C}^{\frac{\dim G}{2}}$

if \mathfrak{X}_0 is an equiv cpt of \mathbb{C}^n , then $\text{Pic}(\mathfrak{X}_0)$ is generated by boundary div. & so is $\text{Pic}(\bar{G})$

boundary div of $\bar{G} \longleftrightarrow$ simple roots

so $\text{Pic}(\bar{G}) = \text{root lattice}$

general fact \longrightarrow
Weight lattice

\mathcal{W}
equality
 \Updownarrow

$G = G_2, F_4$ or E_8

Summarize:

if $G \neq A, G_2, F_4, E_8$

then \bar{G} is rigid under Fano deform.

Case of G_2 F_4 E_8

$T > T' = 2\text{-dim torus}$ corresp. $\mathfrak{h}_{\omega_i^\vee} \oplus \mathfrak{h}_{\omega_j^\vee}$
 coweights

$$\mathcal{X} \supset G > T'$$

$$\pi \downarrow$$

$$\Delta$$

$$Y = \mathcal{X}^{T'}$$

$$\nu \downarrow$$

$$\Delta$$

Y_t is a toric surface $t \neq 0$

Y_0 is an equiv. comp. of \mathbb{C}^2

classified by Hassett-Tschinkel

$Y_t = \overline{T'}$ has Picard number 10, 6, 6
 $G_2 \quad F_4 \quad E_8$

\exists subgp $W' \subset W(G)$ of order 12, 8, 8 acting on $\partial \overline{T'}$ with 2 orbits, NOT possible for Y_0 .

A construction of \overline{C}_ℓ

(W_ℓ, ω_ℓ) sympl. vect. space of $\dim 2\ell$

$LG(W_1 \oplus W_2) = \text{Lagr. Gras. } \hookrightarrow \iota = (1, -1)$

$Z = LG(W_1 \oplus W_2) /_{\langle \iota \rangle} \supset Z_1 \supset Z_2 \supset \dots \supset Z_\ell$

$Z_i = \{[V] \in LG \mid \dim F \cap W_1 \geq i\}$

$Z_\ell = LG(W_1) \times LG(W_2)$

$C_\ell = PSp(2\ell) \xrightarrow{\cong} Z \backslash Z_1$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \overline{C}_\ell & \xrightarrow{\phi} & Z \end{array}$$

ϕ extends to $C_\ell \times C_\ell$ -equiv. morphism

& ϕ is a successive blowups along Z_i

A construction of \overline{B}_ℓ

$(W_i, \langle \cdot, \cdot \rangle)$ orth. space of dim $2\ell+1$

$OG(W_1 \oplus W_2)$ spinor variety

U

$OG(1) \supset OG(2) \supset \cdots \supset OG(\ell)$

$$OG(k) = \{[V] \in OG \mid \dim V \cap W_i \geq k\}$$

$$B_\ell = SO_{2\ell+1} \xrightarrow{\cong} OG \backslash OG(1)$$

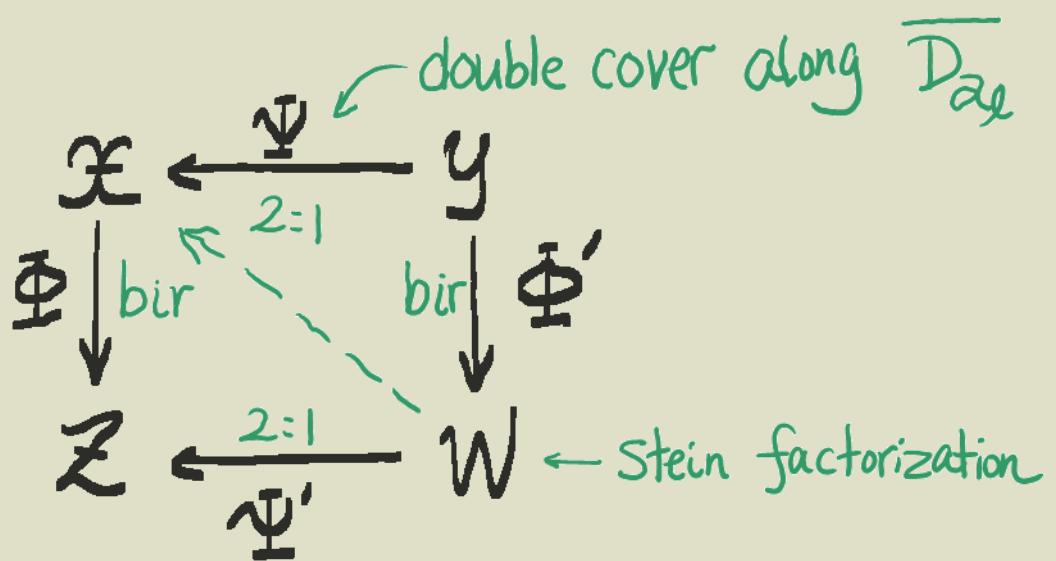
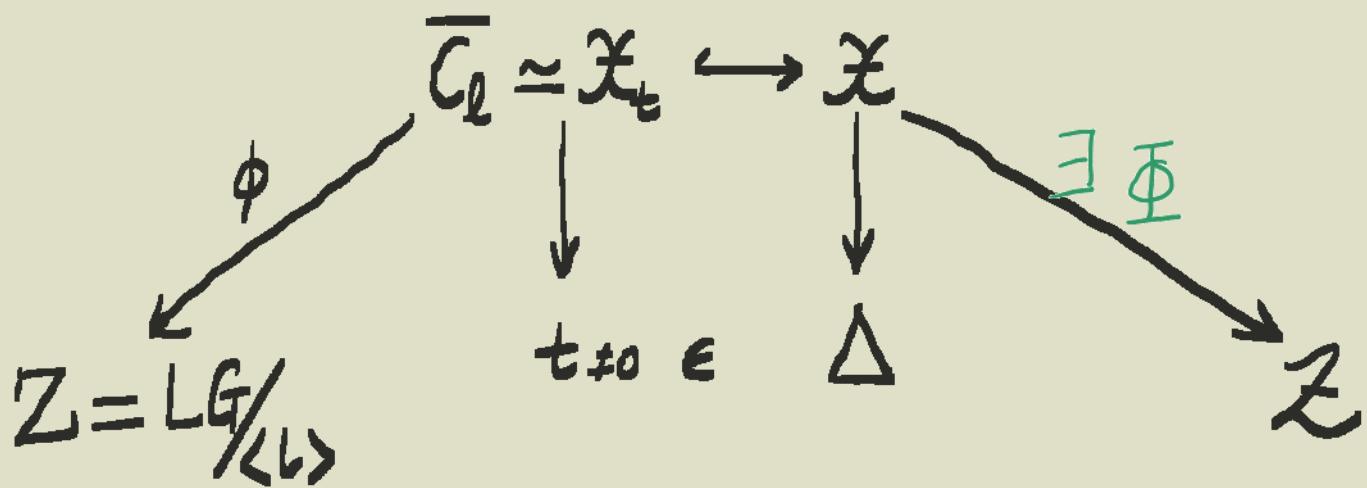
$$\begin{array}{ccc} \downarrow & & \downarrow \\ \overline{B}_\ell & \xrightarrow[\phi]{\quad} & OG(W_1 \oplus W_2) \end{array}$$

ϕ extends to a $B_\ell \times B_\ell$ -equiv. morphism

which is a successive blowup along $OG(k)$

(Similar result holds for \overline{D}_ℓ)

Rigidity of \overline{C}_2



W is a family of LG i.e. $W_0 \simeq LG$

$$Z_0 \simeq LG/\langle \iota \rangle \supset C_2 \times C_2$$

$$\begin{array}{c}
 \mathfrak{X}_0 \\
 \downarrow \text{bir.} \\
 Z_0
 \end{array}$$

$C_2 \times C_2$ -action lifts to \mathfrak{X}_0 , making it spherical.
Show with same colored fan.

Final Questions

- (1) Is \overline{G} rigid under complex deform?
- (2) Rigidity of wonderful symm. var

locally rigid by Bier-Brion

VMRT is studied by Brion-Perrin

- (3) Rigidity for loc. rigid $G/\!\!/ H^s$

Such $G/\!\!/ H^s$ are classified in [BFM]

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Thanks!